ONE CLASS OF MOTIONS OF A FLUID ELLIPSOID

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The motion of a finite mass of an ideal fluid completely bounded by a free surface in which the velocity vector is a linear function of the coordinates is discussed. Surface tension and body forces are absent. One should note that motion with a linear velocity field is also possible in the more general case in which the fluid particles may be attracted according to Newton's law (for example, see [1]). The region occupied by the fluid necessarily has the shape of an ellipsoid. The motion of a fluid ellipsoid in the absence of body forces was investigated by L. V. Ovsyannikov in [2], in which in particular exact solutions were found which describe the deformation of an ellipsoid of revolution and instability of potential motions relative to vortical perturbations is shown. For all the solutions constructed in [2], the directions of the constant momentum and circulation vectors coincide with the direction of one of the axes of the ellipsoid. All motions possessing this property are investigated in this paper. The instability of the exact solutions obtained in [2] relative to perturbations which conserve momentum and circulation is proved. Under certain additional conditions exact asymptotic changes in the lengths of the semiaxes of the ellipsoid are obtained at large values of the time.

1. Statement of the Problem. At t = 0 let the directions of the coordinate axes coincide with the directions of the ellipsoid axes, and let the origin of coordinates coincide with the center of mass of the ellipsoid. Then for the class of motions under discussion the velocity field u and pressure field p can be represented in the form [2-4]

$$u(x, t) = A'(t)A^{-1}x,\ p(x, t) = -(\rho/2)a(t)(xA^{-1}A^{*-1}x - 2c).$$

Here ρ is the fluid density, c is an arbitrary positive constant, and A(t) and a(t) are an unknown matrix and scalar function which satisfy the system of equations

$$A'' = a(t)A^{*-1}; \tag{1.1}$$

$$\det A = n^3 \tag{1.2}$$

and the initial data

$$A(0) = N,\quad A'(0) = A'_0 \tag{1.3}$$

where N is a diagonal positive matrix and A'_0 is an arbitrary matrix satisfying the matching conditions with Eq. (1.2) $\det N = n^3$, $\mathrm{Sp}(N^{-1}A'_0) = 0$, a prime denotes differentiation with respect to the time, and * denotes transposition of the matrix.

The equation of a free surface is of the form $xA^{-1}A^{*-1}x = 2c$.

We note that the transformation $t \to n^2t$, $A \to nA$ leaves Eq. (1.1) unchanged, and Eq. (1.2) reduces to the form det A = 1. It is assumed everywhere below that this transformation has already been made.

The unique solvability of the problem (1.1)-(1.3) for all t > 0 was proved in [3]. Some sufficient conditions for the unbounded increase of one of the ellipsoid semiaxes have been established in [4], and it has been shown that the solution of the system (1.1) and (1.2) describes the motion of a point along a geodesic on the surface $\det A = n^3$ in R^9 at a constant velocity, which has permitted finding a new class of exact solutions in this problem with a matrix A linear in t.

Following [2, 3], one can reduce the system (1.1) and (1.2) to the normal form:

$$A'' = [\mathrm{Sp}(A^{-1}A')^2/\mathrm{Sp}(A^{-1}A^{*-1})]A^{*-1}. \tag{1.4}$$

Equation (1.2) is the zeroth-order integral of the system (1.4). In addition this system has seven more first-order integrals corresponding to the physical laws of energy conservation

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$$(1/2)\mathrm{Sp}(A'A'^*) = E, \tag{1.5}$$

momentum conservation

$$A'A^* - AA'^* = \mathbf{C}, \tag{1.6}$$

and circulation conservation

$$A'^*A - A^*A' = \mathbf{L}, \tag{1.7}$$

where E is a positive constant and **C** and **L** are constant antisymmetric matrices.

Now let the matrices **C** and **L** have the form

$$\mathbf{C} = \begin{pmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & l & 0 \\ -l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.8}$$

i.e., the direction of the momentum and circulation vectors coincides with the direction of the x_3 axis. In this case the solution of the problem (1.4) and (1.3) is of the form

$$A(t) = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_5 \end{pmatrix}.$$

If one makes the variable replacement $A(t) = Q_1 Z Q_2$, where $Z = \mathrm{diag}(z_1, z_2, z_3)$,

$$Q_1 = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Eq. (1.2) is written in the form

$$z_1 z_2 z_3 = 1, \tag{1.9}$$

and the system (1.1) will take the form

$$\begin{aligned} z_1'' &= a(t)\, z_1^{-1} + z_1(\varphi'^2 + \psi'^2) - 2z_2\varphi'\psi', \\ z_2'' &= a(t)\, z_2^{-1} + z_2(\varphi'^2 + \psi'^2) - 2z_1\varphi'\psi', \\ z_3'' &= a(t)\, z_3^{-1}; \end{aligned} \tag{1.10}$$

$$\begin{aligned} \varphi'' z_1 - \psi'' z_2 + 2\varphi' z_1' - 2\psi' z_2' &= 0, \\ \varphi'' z_2 - \psi'' z_1 + 2\varphi' z_2' - 2\psi' z_1' &= 0 \end{aligned} \tag{1.11}$$

after multiplication by Q_1^{-1} on the left and by Q_2^{-1} on the right. The identities

$$\begin{aligned} 2\psi' z_1 z_2 - \varphi'\left(z_1^2 + z_2^2\right) &= c, \\ 2\varphi' z_1 z_2 - \psi'\left(z_1^2 + z_2^2\right) &= l, \end{aligned} \tag{1.12}$$

which are identical to the integrals (1.6) and (1.7) if the matrices **L** and **C** are of the form (1.8), follow from Eqs. (1.11).

If one expresses φ' and ψ' from (1.12) and substitutes into (1.10),

$$\begin{aligned} z_1'' &= a(t)\, z_1^{-1} + \left[(c^2 + l^2)\, z_1\left(z_1^4 - 3z_2^4 + 2z_1^2 z_2^2\right) - 2cl z_2\left(z_2^4 - 3z_1^4 + 2z_1^2 z_2^2\right)\right]\left(z_1^2 - z_2^2\right)^{-4}, \\ z_2'' &= a(t)\, z_2^{-1} + \left[(c^2 + l^2)\, z_2\left(z_2^4 - 3z_1^4 + 2z_1^2 z_2^2\right) - 2cl z_1\left(z_1^4 - 3z_2^4 + 2z_1^2 z_2^2\right)\right]\left(z_1^2 - z_2^2\right)^{-4}, \quad z_3'' = a(t)\, z_3^{-1} \end{aligned} \tag{1.13}$$

is obtained. One can write system (1.13) in the more compact form

$$\begin{aligned} z_1'' &= a(t)\, z_1^{-1} + c_1^2 (z_1 + z_2)^{-3} + c_2^2 (z_1 - z_2)^{-3}, \\ z_2'' &= a(t)\, z_2^{-1} + c_1^2 (z_1 + z_2)^{-3} - c_2^2 (z_1 - z_2)^{-3}, \\ z_3'' &= a(t)\, z_3^{-1}, \end{aligned} \tag{1.14}$$

having set $c_1 = (l - c)/\sqrt{2}$ and $c_2 = (l + c)/\sqrt{2}$.

If one differentiates (1.9) twice and uses (1.14), one can obtain an expression for $a(t)$ in terms of z_i and z_i':

$$a(t) = \left(\sum_{i=1}^{3} \frac{1}{z_i^2}\right)^{-1} \left[\sum_{i=1}^{3} \frac{z_i'^2}{z_i^2} - \frac{c_1^2}{z_1 z_2 (z_1 + z_2)^2} + \frac{c_2^2}{z_1 z_2 (z_1 - z_2)^2}\right]. \tag{1.15}$$

The energy integral (1.5) in terms of c_i and z_i is of the form

$$\sum_{i=1}^{3} z_i'^2 + \frac{c_1^2}{(z_1 + z_2)^2} + \frac{c_2^2}{(z_1 - z_2)^2} = E. \tag{1.16}$$

It was shown in [4] that for any solution of the system (1.14) and (1.15) at least one of the following equations is true:

$$\lim_{t\to\infty} z_i(t) = \infty. \tag{1.17}$$

When $i = 3$, Eq. (1.17) is possible only if $c_1 = c_2 = 0$. One can assume without restricting generality that $\lim\limits_{t\to\infty} z_1(t) = \infty$ is satisfied.

Taking z_1 as the new independent variable, using the identity (1.16), and expressing z_3 from (1.9), one can reduce the system (1.14) and (1.15) to a single nonautonomic second-order equation:

$$\begin{aligned} d^2z_2/dz_1^2 = {} & 2z_1^{-1}z_2^{-1}\left(z_1^2 + z_2^2 + z_1^4 z_2^4\right)^{-1}\left[z_1 z_2^2 + \left(z_1^2 z_2 - z_2^3\right) dz_2/dz_1 + \left(z_1^3 - \right.\right. \\ & \left.\left. - z_1 z_2^2\right)(dz_2/dz_1)^2 - z_1^2 z_2 (dz_2/dz_1)^3\right] + E^{-1}\left[1 + z_1^{-4}z_2^{-2} + 2z_1^{-3}z_2^{-3} dz_2/dz_1 + \right. \\ & \left. + \left(1 + z_1^{-2}z_2^{-4}\right)(dz_2/dz_1)^2\right]\left\{c_1^2\left(z_1 + z_2^{-2}\left(1 - dz_2/dz_1\right)\left[(z_1 + z_2)^{-1} - z_1 z_2\left(z_1^2 + z_2^2 + \right.\right.\right.\right. \\ & \left.\left.\left. + z_1^4 z_2^4\right)^{-1}\right] + c_2^2\left(z_1 - z_2\right)^{-2}\left[(z_1 - z_2)^{-1}(1 + dz_2/dz_1) + z_1 z_2\left(z_1^2 + z_2^2 + z_1^4 z_2^4\right)^{-1}(1 - dz_2/dz_1)\right]\right\}. \end{aligned} \tag{1.18}$$

2. Asymptote of the Solutions at $t \to \infty$. First we shall consider solutions for which the matrix A is diagonal, $A = \mathrm{diag}(z_1, z_2, z_3)$. It has been shown in [1] that the condition $L = C = 0$ is necessary and sufficient for the matrix A to be diagonal in some coordinate system. The quantities z_1, z_2, and z_3 should evidently satisfy the system (1.14) and (1.15) with $c_1 = c_2 = 0$. It is not difficult to see that in this case the right-hand sides of Eq. (1.14) are non-negative; therefore

$$z_i'' \geqslant 0, \qquad i = 1, 2, 3, \tag{2.1}$$

whence we conclude that if $z_i'(t^*) > 0$ for some t^*, the inequalities

$$z_i(t^*) + (t - t^*)\, z_i'(t^*) \leqslant z_i(t) \leqslant z_i(t^*) + (t - t^*)\sqrt{E} \tag{2.2}$$

are satisfied for all $t > t^*$. The upper limit in (2.2) follows from (1.16). It also follows from (2.1) that if $z_i(t)$ is bounded, a value of c exists such that

$$-ct^{-1} < z_i'(t) < 0. \tag{2.3}$$

Evaluating the right-hand side in (1.14) using (2.2) and (2.3), one can obtain a detailed asymptote of $z_i(t)$ at large t.

First let us consider the regime of motion in which as $t \to \infty$ two ellipsoid semiaxes z_1 and z_2 increase without limit. A sufficient condition for its realization is: $z_1'(0) > 0$ and $z_2'(0) > 0$. The existence of b_i and d_i such that

$$b_i t \leqslant z_i(t) \leqslant d_i t, \quad b_i \leqslant z_i'(t) \leqslant d_i \tag{2.4}$$

follows from the bound (2.2) at sufficiently large t. Using (2.4), we shall estimate the value of $a(t)$ at large t. By virtue of (1.9) we have

$$\begin{aligned} a(t) = {} & 2\left(z_1^{-2} + z_2^{-2} + z_1^2 z_2^2\right)^{-1}\left(z_1'^2 z_1^{-2} + z_2'^2 z_2^{-2} + z_1' z_2' z_1^{-1} z_2^{-1}\right) \leqslant 2\left(b_1^{-2}t^{-2} + \right. \\ & \left. + b_2^{-2}t^{-2} + d_1^2 d_2^2 t^4\right)^{-1}\left(d_1^2 b_1^{-2} t^{-2} + d_2^2 b_2^{-2} t^{-2} + d_1 d_2 b_1^{-1} b_2^{-1} t^{-2}\right) \leqslant \gamma t^{-6}. \end{aligned}$$

The lower bound is obtained just as exactly. Finally, we have

$$\delta t^{-6} \leqslant a(t) \leqslant \gamma t^{-6}. \tag{2.5}$$

Substituting the bounds (2.4) and (2.5) into Eqs. (1.14) and integrating the differential inequalities obtained, we conclude that there exist constants $\alpha_i > 0$, β_i, $\gamma_i > 0$, $\delta_i > 0$ such that at large t

$$\alpha_i t + \beta_i + \gamma_i t^{-5} \leqslant z_i(t) \leqslant \alpha_i t + \beta_i + \delta_i t^{-5} \qquad (2.6)$$

is satisfied. The bound $z_3(t)$ follows from the bounds (2.6) and the integral (1.9) and has the form $z_3(t) = O(t^{-2})$ at $t \to \infty$.

In addition to what has just been described, two other regimes of expansion of the ellipsoid are also possible at $t \to \infty$.

1. One axis z_1 tends to infinity as $t \to \infty$, and another axis z_3 tends to zero, and the third axis z_2 tends to some constant α_3 which is different from zero. There exist positive α_1, β_1, γ_1, γ_2, γ_3 such that

$$\begin{gathered} \alpha_1 t + \beta_1 + \gamma_1 t^{-3} \leqslant z_1(t) \leqslant \alpha_1 t + \beta_1 + \gamma_2 t^{-3}, \\ \alpha_3 \leqslant z_2 \leqslant \alpha_3 + \gamma_3 t^{-1}. \end{gathered} \qquad (2.7)$$

2. The one semiaxis z_1 tends to infinity as $t \to \infty$, and the other two tend to zero. In this case the following inequalities are satisfied:

$$\alpha_1 t + \beta_1 \leqslant z_1(t) \leqslant \alpha_1 t + \beta_1 + \gamma_1 t^{-1} \qquad (2.8)$$

with some β_i, $\alpha_i > 0$, $\gamma_i > 0$.

Now we shall consider the overall system (1.14) in the case $c_1^2 \leqslant c_2^2$. It is evident from formula (1.15) that in this case $a(t) > 0$, and consequently the inequalities $z_1'' \geqslant 0$ and $z_3'' \geqslant 0$ are satisfied if $z_1(0) > z_2(0)$.

Proceeding just as in the case $c_1 = c_2 = 0$, one can show that bounds of the type (2.8) are satisfied for an infinitely increasing ellipsoid semiaxis.

3. Stability of the Exact Solutions. Equation (1.18) with $c_2 = 0$ has the solution $z_2 = z_1$, and with $c_1 = c_2 = 0$ it has in addition the solution $z_2 = z_1^{-1/2}$. The corresponding solutions of the system (1.1) and (1.2) describe the deformation of a fluid ellipsoid of revolution; in the first case two semiaxes of this ellipsoid increase without limit as $t \to \infty$, and in the second case — one. Both of these solutions were found by L. V. Ovsyannikov. It was shown by him that the second solution, which describes the potential motion of a fluid, is unstable with respect to vortical perturbations (for Eq. (1.18) this means that its solution $z_2 = z_1^{-1/2}$ is unstable with respect to perturbations of the parameter c_1); as small vortical perturbations as desired lead to the fact that as $t \to \infty$ two ellipsoid semiaxes increase without limit and not one, as in the original solution. It is shown below that as small potential perturbations of this motion as desired lead to the very same result. Instability of the first motion with respect to perturbations which conserve momentum and circulation has also been shown, i.e., instability of the exact solutions of Eqs. (1.18) $z_2 = z_1^{-1/2}$ and $z_2 = z_1$ with respect to perturbations of the initial data has been proved.

Let $c_2 = 0$. Let us introduce the notation $z_1 = y$ and consider the Cauchy problem for Eq. (1.18)

$$z_2(y_0) = y_0, \quad d\dot{z}_2/dy = 1 + z_0 \quad \text{at} \quad y = y_0.$$

The solution $z_2(y)$ is sought in the form $z_2(y) = y + z(y)$. Equation (1.18) takes the form

$$\begin{gathered} \ddot{z} = -2y^{-1}(y+z)^{-1}[2y^2 + 2yz + z^2 + y^4(y+z)^4]^{-1}[3y^2 z + \\ + 3yz^2 + z^3 + \dot{z}(3y^3 + 9y^2 z + 5yz^2 + z^3) + \dot{z}^2(3y^3 + 5y^2 z + yz^2) + \\ + \dot{z}^3(y^3 + y^2 z)] - c\dot{z}(2y+z)^{-2}\{(2y+z)^{-1} - (y^2 + yz)[2y^2 + \\ + 2yz + z^2 + y^4(y+z)^4]^{-1}\}[1 + y^{-4}(y+z)^{-2} + 2(1+\dot{z})y^{-3}(y+z)^{-3} + (1+\dot{z})^2 y^{-2}(y+z)^{-4} + (1+\dot{z})^2], \end{gathered} \qquad (3.1)$$

in the variables y and z, where $\dot{z} = dz/dy$, $\ddot{z} = d^2z/dy^2$, $c = c_1^2/E$. Now let $y_0 \geqslant 2$ and the inequality

$$\max(|z|, |\dot{z}|/y) \leqslant 1/16 \qquad (3,2)$$

be satisfied on some interval (y_0, y_1). Multiplying both sides of Eq. (3.1) by $2\dot{z}$ and evaluating the right-hand side under the conditions (3.2), one can obtain

$$d(\dot{z}^2)/dy \leqslant (3/4)y^{-7},$$

whence it follows that

$$\dot{z}^2 \leqslant \dot{z}_0^2 + 2^{-3}y_0^{-6} \leqslant \dot{z}_0^2 + 2^{-9}$$

and this means condtion (3.2) is satisfied for all y only if

$$\dot{z}_0^2 < 2^{-9}. \qquad (3.3)$$

Now let (3.3) be satisfied. We multiply both sides of Eq. (3.1) by $\dot{z}y^{-8}(y + z)[2y^2 + 2yz + z^2 + y^4(y + z)^4]$ and evaluate the right-hand side with the help of the inequality (3.2). After simple transformations we obtain that if $y > y_0 \geqslant \max(2, c^{1/3})$,

$$\frac{d}{dy}\left\{\dot{z}^2\frac{y+z}{y^8}[2y^2+2yz+z^2+y^4(y+z)^4]-\frac{z^2}{2y}\right\} \geqslant 0$$

and consequently,

$$\dot{z}^2 y^{-8}(y+z)[2y^2+2yz+z^2+y^4(y+z)^4]-z^2y^{-1}/2 \geqslant \dot{z}_0^2\left(y_0+2y_0^{-5}\right). \qquad (3.4)$$

Using the inequalities (3.2) and (3.4), we obtain

$$\dot{z}^2 \geqslant \dot{z}_0^2 y_0/(4y),$$

from which

$$|z| \geqslant |\dot{z}_0|\, y_0^{1/2}\left(y^{1/2}-y_0^{1/2}\right)$$

follows.

Thus it has been shown that the solution $z_2 = z_1$ of Eq. (1.18) is unstable when $z_1 \geqslant \max(2, c^{1/3})$.

The following assertion is also true: For any positive $\alpha < 1$ values of y_2 and δ are found such that the solution of the Cauchy problem $z(y_0) = 0$, $\dot{z}(y_0) = \dot{z}_0$ for Eq. (3.1) satisfies the inequality

$$|z(y)| \geqslant |\dot{z}_0|\, y_0^{1-\alpha}\left(y^{\alpha}-y_0^{\alpha}\right)$$

for all $y > y_0$ only if $|\dot{z}_0| < \delta$, $y_0 > y_1$.

For proof of this assertion it is necessary to multiply both sides of Eq. (3.1) by $\dot{z}(y + z)^{-2\alpha-7}[2y^2 + 2yz + z^2 + y^4(y + z)^4]$ and perform discussions similar to those above in the case $\alpha = 1/2$.

Now let $c_1 = c_2 = 0$. In this case the inequalities (2.6) are satisfied for the ellipsoid axes z_1 and z_2 which increase without limit. By virtue of what has been proved above, if for some t_0

$$z_1(t_0) = z_2(t_0), \quad z_1'(t_0) \neq z_2'(t_0),$$

$\alpha_1 \neq \alpha_2$ in the inequalities (2.6), and consequently for large t the quantity $|z| = |z_1 - z_2|$ increases as a linear function of t.

Equation (1.18) with $c_1 = c_2 = 0$ has, in addition to what has been discussed above, the exact solution $z_2 = z_1^{-1/2}$. It will be shown below that the solution of the system (1.14) and (1.15) corresponding to it is the only solution of this system in the case $c_1 = c_2 = 0$, in which the lengths of two ellipsoid semiaxes tend to zero at $t \to \infty$.

We note that with $c_1 = c_2 = 0$ the inequalities (2.1) and (2.2) and the energy integral

$$z_1'^2\left(1+z_1^{-4}z_2^{-2}\right)+2z_1'z_2'z_1^{-3}z_2^{-3}+z_2'^2\left(1+z_1^{-2}z_2^{-4}\right)=E$$

occur for the solution of the problem (1.14) and (1.9). It follows from this that if $\dot{z}_2(y_1) > 0$ for some y_1, the inequality

$$\dot{z}_2(y) \geqslant \dot{z}_2(y_1)\left[1+y_1^{-4}z_2^{-2}(y_1)+2\dot{z}_2(y_1)\,y_1^{-3}z_2^{-3}(y_1)+\dot{z}_2^2(y_1)+\dot{z}_2^2(y_1)\,y_1^{-2}z_2^{-4}(y_1)\right]^{-1/2} \qquad (3.5)$$

is satisfied for all $y > y_1$.

Now let us assume that there exists a solution of Eq. (1.18) with $c_1 = c_2 = 0$ other than $z_2 = z_1^{-1/2}$, such that

$$\lim_{y\to\infty} z_2(y) = \lim_{y\to\infty}[yz_2(y)]^{-1} = 0. \qquad (3.6)$$

By virtue of the property noted above of the solution (3.5), the semiaxes z_2 and $z_3 = (yz_2)^{-1}$ should decline as y increases:

$$\dot{z}_2 < 0, \quad y^{-1} + \dot{z}_2 z_2^{-1} > 0. \tag{3.7}$$

It follows from (3.6) that for any $\varepsilon > 0$ a $y_2(\varepsilon)$ is found such that when $y > y_2(\varepsilon)$ $(\varepsilon y)^{-1} < z_2 \leqslant \varepsilon$. We shall assume that there exists a $y_0 > y_2(0.04)$ such that

$$2\dot{z}_2(y_0) > -y_0^{-3/2}. \tag{3.8}$$

Let us consider for Eq. (1.18) the Cauchy problem with initial data at the point y_0:

$$z_2(y_0) = z_0 + y_0^{-1/2}, \quad \dot{z}_2(y_0) = \dot{z}_0 - y_0^{-3/2}/2.$$

By virtue of the inequalities (3.7) and (3.8),

$$0 < \dot{z}_0 < y_0^{-3/2}/2. \tag{3.9}$$

Let $z = z_2 - y^{-1/2}$. It follows from the continuity of the solution of (1.18) that (3.9) is satisfied on some interval (y_0, y_1). Multiplying both sides of the equation by $2\dot{z}y$, adding the quantity $\dot{z}^2 + z^2y^{-2}/9 - 2\dot{z}zy^{-1}/9 - 2zy^{-3/2}/3$ to it, and evaluating the right-hand side of the equation obtained, one can show that the inequality

$$d(\dot{z}y - z^2y^{-1}/9)/dy \geqslant 0 \tag{3.10}$$

is satisfied on the interval (y_0, y_1), whence it follows that if

$$\dot{z}_0 > |z_0| y_0^{-1}/3 \tag{3.11}$$

a y_1 exists such that $2\dot{z}(y_1) > y_1^{-3/2}$. Thus instability of the solution $z_2 = y^{-1/2}$ of Eq. (1.18) has been proven.

It is sufficient for proof of the fact that $z_2 = y^{-1/2}$ is the only solution of (1.18) which satisfies (3.6) in the case $c_1 = c_2 = 0$ to show now that for any solution of (1.18) with the properties (3.6) and (3.7) other than $z = y^{-1/2}$ a $y_0 > y_2$ (0.04) is found such that for $y = y_0$ the inequalities (3.9) and (3.11) are satisfied for one of the functions z, $\bar{z} = (yz_2)^{-1} - y^{-1/2}$. It is evident that $\bar{z} = z_3 - y^{-1/2}$ satisfies the very same equation as does z if $c_1 = c_2 = 0$ and all the assertions proved above for z are true for $\bar{z}$.

For any continuous function $z_2(y)$ one of the three following assertions is true:

1) For any y_1 a $y_0 > y_1$ is found such that $z_2(y_0) = y_0^{-1/2}$;

2) there exists a y_1 such that $z_2 < y^{-1/2}$ for all $y > y_1$; and

3) there exists a y_1 such that $z_2 > y^{-1/2}$ for all $y > y_1$.

In case 1 condition (3.11) is satisfied if $z(y_0) > 0$, since $z(y_0) = 0$, and $|\dot{z}(y_0)| > 0$ since in the opposite case the solution would coincide with $z_2 = y^{-1/2}$. If $\dot{z}(y_0) < 0$, the inequalities (3.9) and (3.11) are satisfied for $\bar{z}$.

Now let us consider case 2. We shall assume that for all $y > y_1 > y_2$ (0.04)

$$\dot{z} \leqslant zy^{-1}/3 \tag{3.12}$$

is satisfied; then integrating (3.12) and taking account of the fact that $z < 0$, we obtain

$$z(y) \leqslant z(y_1) y_1^{1/3} y^{-1/3},$$

which contradicts the positive nature of the quantity $z_2 = z + y^{-1/2}$ at sufficiently large y. Thus it has been shown that there exists a $y_0 > y_2$ (0.04) such that

$$\dot{z}(y_0) > z(y_0) y_0^{-1}/3,$$

from which (3.11) and the first of the inequalities (3.9) follow. The second inequality (3.9) follows from the assumption $\dot{z}_2 < 0$.

Case 3 reduces to case 2 if one considers z_3 instead of z_2.

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LITERATURE CITED

1. S. Chandrasekhar, Ellipsoidal Figures of Equilibrium, Yale Univ. Press (1970).
2. L. V. Ovsyannikov, "A class of nonsteady motions of an incompressible fluid," in: Transactions of the Fifth Session of the Scientific Council on the National Economic Use of Explosion [in Russian], ILIM, Frunze (1965).
3. V. I. Nalimov and V. V. Pukhnachev, Nonsteady Motions of an Ideal Fluid with a Free Boundary [in Russian], NGU, Novosibirsk (1975).

4. O. M. Lavrent'eva, "The motion of a fluid ellipsoid," Dokl. Akad. Nauk SSSR, 253, No. 4 (1980).

EXACT SOLUTION OF THE THREE-DIMENSIONAL PROBLEM OF IDEAL PLASTICITY

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Let $r\theta z$ be a cylindrical coordinate system, σ_r, σ_θ, σ_z, $\tau_{r\theta}$, τ_{rz}, $\tau_{\theta z}$ be the components of the stress tensor, u, v, and w be the components of the velocity vector, and k be the yield stress under pure shear.

The equations of ideal plasticity with the von Mises yield condition are of the form

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0,$$
$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0,$$
$$\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0,$$
$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\left(\tau_{r\theta}^2+\tau_{rz}^2+\tau_{\theta z}^2\right)=6k^2, \tag{1}$$
$$\lambda\frac{\partial u}{\partial r}=2\sigma_r-\sigma_\theta-\sigma_z,\quad \lambda\left(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right)=2\sigma_\theta-\sigma_z-\sigma_r,$$
$$\lambda\frac{\partial w}{\partial z}=2\sigma_z-\sigma_r-\sigma_\theta,\quad \lambda\left(\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z}\right)=2\tau_{\theta z},$$
$$\lambda\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=2\tau_{rz},\quad \lambda\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)=2\tau_{r\theta},$$
$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial\theta}+r\frac{\partial w}{\partial z}=0,$$
$$\sigma_r+\sigma_\theta+\sigma_z=3p.$$

We shall assume that

$$\tau_{rz}=\tau_{r\theta}=0. \tag{2}$$

We shall seek the solution of Eqs. (1) in the form

$$u=u^*(r)\operatorname{sh}\xi,\ v=v^*(r)\operatorname{ch}\xi,\ w=w^*(r)\operatorname{ch}\xi,\ p=p(r),\ \xi=z+\beta\theta, \tag{3}$$

where u*, v*, w* are functions only of r and β is an arbitrary constant. Then we obtain from the incompressibility condition and Eqs. (2) a system of ordinary differential equations for determination of the functions u*, v*, w*:

$$u^*+\frac{dw^*}{dr}=0,\quad r\frac{d}{dr}\left(\frac{v^*}{r}\right)+\frac{\beta}{r}u^*=0,\quad \frac{d}{dr}(ru^*)+\beta v^*+rw^*=0, \tag{4}$$

and the equation

$$d\sigma_r/dr+(\sigma_r-\sigma_\theta)/r=0 \tag{5}$$

remains for the determination of the pressure p. The system of equations (4) reduces to the Bessel equation

$$r^2u^{*\prime\prime}+ru^{*\prime}-(r^2+\beta^2+1)u^*=0.$$

The solution of this equation is of the form

$$u^*=C_1I_\nu(r)+C_2K_\nu(r),\ \nu=\sqrt{\beta^2+1}, \tag{6}$$

where I_ν are the Bessel functions of imaginary argument, K_ν is the MacDonald function, and C_1 and C_2 are arbitrary constants. If one sets $C_2 = 0$ in (6), the velocity field is of the form

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